Ph 1b Recitation Notes
Section 7

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1 Electrostatics in Materials

1.1 The Picture

So far, we have only considered electrostatics in vacuum, or in materials that have a set charge distribution. But how do we get these charge distributions in the first place? Fundamentally, charges come from charged particles (ie. electrons, protons, etc.). Since the temperatures of everyday material are fairly low (by particle physics standards), protons bind with neutrons and form nuclei. These nuclei bind with electrons to form atoms, and atoms bind with other atoms to create molecules. The complex interactions of atoms and molecules form the basis of chemistry. Such chemical reactions only involve nuclei and electrons (which are relatively weakly bound to each other). Nuclear reactions (involving much higher temperatures) are required to interact between the constituents (neutrons and protons) of nuclei. Since it requires much less energy to interact chemically, most (classical) materials can be thought of as a bunch of atoms that (potentially) exchange electrons (negative charges). For historical reasons, the convention is actually to think of materials having moving *positive* charges (e.g. moving electron "gaps"). The amount of moving electrons (and the ease to which they move) in a material is called the "electrical conductivity"[1]. A higher conductivity allows more electrons to move and a lower conductivity allows less electrons to move. Naturally, this produces two ideal materials in the extremal cases: those with infinite conductivity and with zero conductivity. We fittingly call these materials "conductors" and "insulators" (also called "dielectrics"), respectively.

Note that this simple picture is only a classical approximation. If you want to do real material science, you have to talk about things like band-gaps caused by quantum effects. If you are interested in this you might want to look into condensed matter physics.

[1]Note that it is ions that move in fluids.

You may ask: How do we move electrons in the first place? In order to move electrons, you need to leave the electrostatic regime and enter electrodynamics (or at least magnetostatics). This is therefore a question for later. For now, we will simply assume that the materials have reached an equilibrium where the charge distribution is constant in time.

1.2 Conductors

Conductors are ideal materials where charges are allowed to freely move in the material. An example of a conductor is any metal. Conductors have some interesting properties (Griffiths, 2017; Purcell and Morin, 2013):

1. $\mathbf{E} = \mathbf{0}$ inside: If there was an electric field inside, then the charges would move and the system would no longer be in electrostatic equilibrium. The equilibrium state must therefore be one without an electric field inside. This equilibrium happens to be stable since any change to the electric field will move the charges such that a new electric field is created to counter balance the existing one, giving no total electric field inside.

2. $\rho = 0$ inside: This must be true by Gauss' Law if property 1 holds.

3. $V =$ const throughout: The integral over any path inside a conductor is zero by property 1, so the change in the potential over any path inside is zero.

4. All charge is on the surface: There is nowhere else for it to be (property 2).

5. $\mathbf{E} \perp$ surface on boundary: If there was a tangential component of the electric field on the surface, then the charges would move along the surface to counter balance the effect.

A nearby external charge can induce a charge on either side of a conductor. This induction is done through an electric field. In order to keep $\mathbf{E} = \mathbf{0}$ inside of the conductor, there needs to be an electric field on either side of the conductor in such a way that it cancels out. Thus, there will be a more negative or positive charge on one side when compared to the other.

Since the electric field is zero inside of the conductor, this can be used to shield outside electric fields (such as EM waves) from entering a region of space. This is called a Faraday cage and can be used, for instance, to shield electrical equipment from radiation from external radio frequency interference.

1.3 Example: Spherical Cavities in a Conductor

This is adapted from problem 2.39 in Griffiths (2017) (see Fig. 2.49 for a visual guide). Consider a conducting sphere of radius R with two hallowed out spherical cavities of radius a and b. In each cavity, there is a central point charge

q_a and q_b, respectively. Find the surface charge densities σ_a, σ_b, σ_R, the field outside of the conductor and in each cavity, the potentials in all locations, and the force on the point charges. What would change if there were more than two cavities with central point charges (e.g. q_c, q_d, etc.)? What about if a third charge q were brought near the conductor (answer this qualitatively and in electrostatic equilibrium)?

The point charges in each cavity will induce surface charges on the boundary of the cavities. These surface charges will need to be such that a Gaussian surface at a radius slightly larger than the cavity gives zero net charge, by Gauss' Law, since $\mathbf{E} = \mathbf{0}$ inside the conductor. So, the total charge on the surface of cavity a and b will be $-q_a$ and $-q_b$, respectively. By spherical symmetry the charge will be distributed evenly along the surface, so the cavities will have surface charge distributions of $\sigma_a = -\frac{q_a}{4\pi a^2}$ and $\sigma_b = -\frac{q_b}{4\pi b^2}$. Since the conductor is initially neutral, it will have have zero total charge. Since there is a charge of $-q_a$ on the surface of cavity a and $-q_b$ on the surface of cavity b, there will need to be a total charge $q_a + q_b$ on the outside surface of the conductor (since the charge is zero inside the conductor). Since the conductor has no net charge, this charge will be distributed evenly along the conductor's outer surface, giving that $\sigma_R = \frac{q_a+q_b}{4\pi R^2}$.

Using a spherical Gaussian surface, we can easily find that $\mathbf{E}(r > R) = \frac{q_a+q_b}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$, where r is measured from the center of the conducting sphere R. The same thing can be done for inside the cavities to find that $\mathbf{E}(r_a < a) = \frac{q_a}{4\pi\epsilon_0 r_a^2}\hat{\mathbf{r}}_a$ and $\mathbf{E}(r_b < b) = \frac{q_b}{4\pi\epsilon_0 r_b^2}\hat{\mathbf{r}}_b$, where r_a and r_b are measured from the point charges q_a and q_b respectively.

Since we have the electric field at all locations, we can easily find the potentials. We will set the potential at $r \to \infty$ to be 0. Then, we can integrate inwards to find that the potential outside of the conductor is $V(r \geq R) = \frac{q_a+q_b}{4\pi\epsilon_0 r}$. Since conductors are equipotentials, then the potential inside the conductor is therefore $V(r_a \geq a, r_b \geq b, r < R) = \frac{q_a+q_b}{4\pi\epsilon_0 R}$. The potentials inside each cavity are therefore $V(r_a < a) = \frac{q_a+q_b}{4\pi\epsilon_0 R} + \frac{q_a}{4\pi\epsilon_0 r_a} - \frac{q_a}{4\pi\epsilon_0 a}$ and $V(r_b < b) = \frac{q_a+q_b}{4\pi\epsilon_0 R} + \frac{q_b}{4\pi\epsilon_0 r_b} - \frac{q_b}{4\pi\epsilon_0 b}$.

Due to spherical symmetry, there is no force on the point charge in the center of each cavity. However, any deviation from the exact center will accelerate the point charges to the cavity surface.

If there were more than two cavities, the same thing would happen near the cavities (i.e. replace a with c, d, etc. to get σ_c, $\mathbf{E}(r_c < c)$, $V(r_c < c)$, etc.), and then the total charge distributed along the outside surface would change to $q_a + q_b \to q_a + q_b + q_c + ...$ (still uniformly distributed). This would accordingly change the charge in the electric field and potential equation outside of the conductor and the normalization of the potentials everywhere, since we set the zero point at infinity.

If a third charge were brought nearby the conductor, then the electric field outside the conductor would change, inducing a charge on the conductor (in addition to the charge induced by the cavities), changing the distribution of charge on the outside of the conductor to no longer be uniform (i.e. σ_R and $\mathbf{E}(r > R)$ would change). However, the charge distributions will not change on the surface of the cavities and neither will the electric field inside the cavities (or in the conductor, of course, since it is zero). All the potentials would also change everywhere, since we set the zero point at infinity.

1.4 Dielectrics

The other simple type of material is an insulator (also called dielectrics). Dielectrics do not allow currents (motion of charges) within. While they cannot move charges across themselves, they can polarize in the presence of an electric field. To give you a picture: Electron clouds in the molecules can be shifted relative to their nuclei, causing an electric field across the molecule. Thus, there can still be a change in an external electric field due to the presence of a dielectric. This effect of is measured by a polarization vector $\mathbf{P}$ where the regular Maxwell equations (i.e. Gauss' Law and derivatives, such as the defintion of potential) change like $\mathbf{E} \to \mathbf{E} + \frac{1}{\epsilon_0}\mathbf{P}$ (and $\mathbf{P} = \mathbf{0}$ in vacuum).

In many materials, this polarization vector is proportional to the electric field $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, where χ_e is a constant called the "electric susceptibility". Such materials are called "linear dielectrics". In such materials, we can simply let $\epsilon_0 \to \epsilon \equiv \epsilon_0 \chi_e$ inside the material and proceed like normal. To formalize this, we typically use $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$ and rewrite Gauss' law as $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$. Note that the vacuum acts like an insulator (dielectric) with $\epsilon = \epsilon_0$.

1.5 Capacitance

If we place two conductors near each other which charge Q and $-Q$, then since they are equipotential across their material, we can easily talk about the potential difference between them. Say they have a potential difference $\Delta V \equiv \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}$. Since $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} \mathrm{d}\tau$, then clearly $E \propto Q$ since doubling ρ doubles Q. Moreover, $\Delta V \propto E$ so therefore

$$C \equiv \frac{Q}{\Delta V} \tag{1}$$

where C is some constant of proportionality. We call C the "capacitance" since it measures the "capacity" to store energy in the system[2]. Such a setup is called a "capacitor".

[2] The energy is really being stored in the electric field.

1.6 Example: Parallel-Plate Capacitor

Consider two parallel thin metal plates that are a distance d apart with charge Q and $-Q$ and area A. Recall that the potential of a uniformly charged plane is $V(z) = \frac{\sigma}{\epsilon_0} z$. Since the plane is uniformly distributed, $\sigma = \frac{Q}{A}$. The potential difference between the plates is therefore $\Delta V = \frac{Q}{\epsilon_0 A} d$. So, the capacitance of the parallel-plate capacitor is

$$C = \frac{A\epsilon_0}{d} \tag{2}$$

Note that if we were to put a dielectric material between the plates, the capacitance would be $C = \frac{A\epsilon}{d}$. Thus, to increase the capacitance of a parallel-plate capacitor you can either increase the area of the capacitor, change the insulator material in between the plates (increase ϵ), or decrease the distance between the plates.

1.7 Example: Coaxial Cable Capacitor

Consider two long coaxial cylinders of length l, radius a and b, and charge Q and $-Q$. We had found that the potential difference between the cylinders is $\Delta V = \frac{Q}{2\pi\epsilon_0 l} \ln\left(\frac{a}{b}\right)$. Therefore, the capacitance between the cylinders is

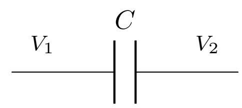
$$C = \frac{2\pi\epsilon_0 l}{\ln a - \ln b} \tag{3}$$

We can therefore increase the capacitance of the coaxial cable capacitor by increasing the length of the cylinders, decreasing the ratio of the inner cylinder radius to the outer cylinder radius, or (as always), by changing the material inside the capacitor.

2 DC Circuits

2.1 Basics

Electrical circuits are connected by wires (conductors) to allow for the flow of current. Since conductors are equipotentials, we can talk about the constant potential V of a wire. In circuit diagrams, wires are written as lines. We have already talked about capacitors, so we will start there. Since the exact value (or gauge) of a potential is arbitrary, we cannot measure the potential of a single wire, but we can measure the potential difference between wires (also called voltage). A voltage V can be measured across a capacitor (e.g. parallel-plate capacitor with some insulator in between the plates) with capacitance C connected by two wires of different potentials. In a circuit diagram, a capacitor is represented with the intuitive symbol

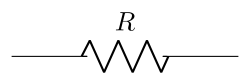


If the wire on the left has potential V_1 and the wire on the right has potential V_2, then the voltage across the capacitor is $\Delta V \equiv V_1 - V_2$. We could then find the charge on the capacitor $Q = C\Delta V$.[3]

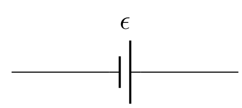
We can also talk about the current $I \equiv \frac{\mathrm{d}Q}{\mathrm{d}t}$, which can be thought of as a flow of charge[4]. We can construct electrical components and connect them with wires of certain potentials. The resistance of the component is measured as $R \equiv \frac{V}{I}$ and can be thought of as the amount of resistance to a flow of charge (e.g. if the current is small and the potential difference is large, the resistance is large since it prevents the flow of charge despite a large voltage). The fact that current and voltage are proportional (by R) is called Ohm's Law[5]:

$$V = IR \tag{4}$$

A component whose purpose is to provide resistance is called a "resistor" and is denoted with the circuit symbol



To actually power a circuit, we need to provide it with something that creates a voltage[6] and close the circuit (otherwise current cannot flow). Typically this is done with, for example a battery that creates a voltage/current through chemical reactions. An ideal battery is just a voltage source. Voltage sources (with open circuit voltage ϵ, also known as the "electromotive force") are denoted with the symbol



[3] Note that typically we drop the deltas for voltage when doing circuits since it is always a potential difference. From here on we will drop them.

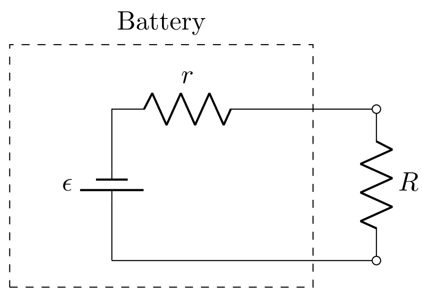
[4] Be careful with this idea though: This is not the speed of power transfer. While a light bulb turned on by a switch across your room happens almost instantaneously, electrons really only move at the crawling speed of only around a centimeter a minute! This will likely be better explained in a course on magnetism.

[5] For now, since we have not yet discussed current in electromagnetism, you will have to take my word that this is true (most of the time). A magnetostatics course will better motivate Ohm's Law. Note that this is not true for certain electronic components: Such components we call "non-ohmic" (e.g. a diode).

[6] Or current, but we will just consider voltage power supplies for now.

where the longer end denotes the positive terminal (higher potential) and the shorter end denotes the negative terminal (lower potential). We conventionally define current as positive charges moving from positive to negative. If this voltage supply is connected to a close circuit, this potential difference will cause an electric field along the wires, allowing the mobile charges in the wires to flow from the positive to the negative terminal.

Since batteries are not ideal, they actually have some small resistance r to them. This resistance, by Ohm's law, means that the voltage across the battery (when connecting it to a close circuit) is really $V = \epsilon - rI$ where I is the current passing through the battery due to the circuit. So, if there is no current flow, the battery is ideal, but if the current is too large, then the battery eventually breaks down and does not deliver any voltage at all[7]. The circuit diagram



describes such a battery, where the dotted region denotes the battery. Note that you can connect the open circles to any circuit. The circuit in between the open circles determines the current I. If we indeed do connect a battery with electromotive force ϵ and resistance r with a resistor R, then the current will happen to be $I = \frac{\epsilon}{r+R}$. We will see how you can determine this yourself in the next section. The power across a resistor R is $P \equiv IV = I^2R$ where I is the current passing through the resistor.

2.2 Kirchhoff Laws

It would be useful to have a systematic way of solving for given quantities (i.e. voltage, current, resistance, capacitance, etc.) in circuits. There are two useful rules that help with this in DC circuits: Kirchhoff's Current Law and Kirchhoff's Voltage Law.

Kirchhoff's Current Law (KCL) states that any closed loop in an electrical circuit, the sum of the potential differences (voltages) of each of the components is zero. Equivalently, this means that

[7] This will usually break your battery.

$$\sum_{k=1}^{n} V_k = 0 \tag{5}$$

with signed voltage values V_k (positive if loop is along the current from the voltage source and negative if not). You can use Ohm's Law to deal with the voltage drop across resistors (where it is negative if the loop is along the current passing through the resistor and positive if it is against the current). This can be used to find how the voltage propagates in an electrical circuit.

Kirchhoff's Voltage Law (KVL) states that any node in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node. Equivalently, this means that

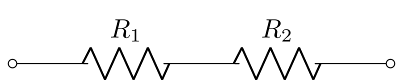
$$\sum_{k=1}^{n} I_k = 0 \tag{6}$$

with signed current values I_k (positive if going into, negative if going out of). This can be used to find how the current propagates in an electrical circuit.

2.3 Equivalent resistance

If you do many of these Kirchhoff Law examples, you might notice that it becomes somewhat tedious for a large amount of resistors. To avoid having to deal with a mess of equations, we can simply a bunch of resistors to one equivalent resistor where we can use Ohm's Law or Kirchhoff's Laws much easier. There are two simple scenarios where you can build equivalent resistances: Those in series and in parallel.

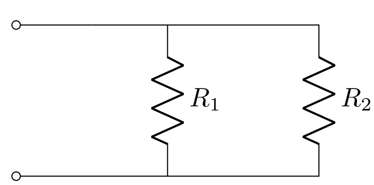
Consider two resistors R_1 and R_2 in *series*



Then, the equivalent resistance (if you were to put a single resistor in between the open circles with same resistance as seen by the rest of the circuit) is

$$R_{\text{eq}} = R_1 + R_2 \tag{7}$$

Consider two resistors R_1 and R_2 in *parallel*.



Then, the equivalent resistance is

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{8}$$

You can also chain many of these equivalent resistances together to solve many (but not all) combinations of resistors. Note that you can prove these equivalent resistance rules directly yourself from Ohm's Law and Kirchhoff Laws if you add a battery to the circuit and close it.

2.4 Equivalent capacitance

Capacitors also have equivalence rules. For capacitors C_1 and C_2 in *series*, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{9}$$

For capacitors C_1 and C_2 in *parallel*, the equivalent capacitance is

$$C_{\text{eq}} = C_1 + C_2 \tag{10}$$

Note that this is sort of the opposite of the equivalent resistance rules.

2.5 Aside: Voltage Divider

KVL and Ohm's Law can be used to combined the voltage divider to find the voltage in one step (for simple circuits with series resistors only and one voltage source). The formula is

$$V_i = \left(\frac{R_i}{R_{\text{eq}}}\right) V_{\text{eq}} \tag{11}$$

where V_i is the voltage across a given resistor i, with resistance R_i, and where V_{eq} is the source voltage for the circuit and R_{eq} is the equivalent resistance.

2.6 Thévenin's Theorem

Using Kirchhoff's laws requires you to re-evaluate the whole circuit even if you change a single resistor. It would be useful to have a rule that allows for a certain part of the circuit to be abstracted to an equivalent circuit (like the equivalent resistance and capacitance). This is called Thévenin's theorem. The equivalent circuit we would like to create is shown in Figure 1 where R_L is some load resistor in the circuit (or equivalent resistance), ϵ_{eq} is the equivalent "Thévenin voltage" and R_{eq} is the equivalent resistance (as seen by the load resistor). Notice that this form looks like an effective battery for the load resistor.

Thévenin's method is

1. Remove the load resistor R_{L} and calculate the open circuit voltage across the gap (called the equivalent Thévenin voltage) ϵ_{eq}.

2. Calculate R_{eq} as the equivalent resistance of the circuit connected to the load if all voltage sources were turned off (shorted).

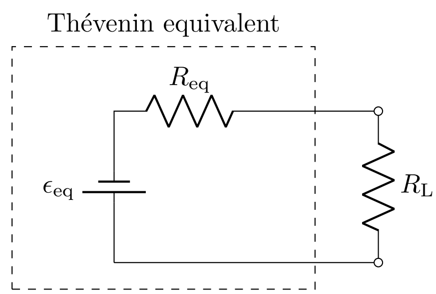


Figure 1: Thévenin's circuit.

In order to use Thévenin's theorem for many different voltage sources, it is useful to use the Superposition Theorem. This principle allows us to calculate the influence of each source of voltage[8] individually on a given component in a circuit. When removing voltage sources, a short-circuit is placed in the circuit[9]. This method works due to the linearity of Kirchoff's current and voltage laws, and ultimately from the superposition principle in EM (where you can simply add potentials or electric field components for different charge distributions).

Note that just like there are voltage sources, there are also current sources. There is in fact an equivalent theorem to Thévenin's Theorem called "Norton's theorem" for current sources with equivalent resistances that are parallel to the load resistance R_L. There are also things that are equivalent to a capacitor but store energy in the magnetic field instead of in the electric field called "inductors".

2.7 Example: Complex Circuit

Consider the circuit in Figure 2. Let $B_1 = 12$ V, $B_2 = 6$ V, $R_1 = 1.5$ kΩ, $R_2 = 3$ kΩ, $R_{\mathrm{L}} = 1$ kΩ, find the currents through all resistors. Replace the dashed line of the circuit by a Thévenin equivalent circuit (Figure 1) with resistance R_{eq} and electromotive force ϵ_{eq}.

[8]Or current source.

[9]Analogously, a break is inserted in place of current sources

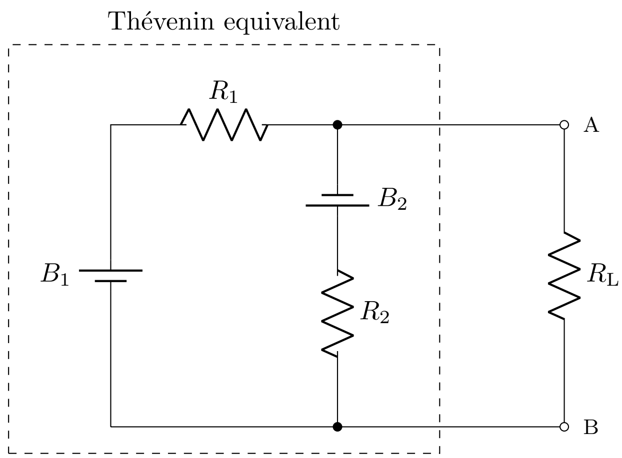


Figure 2: The complex circuit used in this lab to test the Thévenin theorem.

First, we can find that $R_{\mathrm{L}} = 1\ \mathrm{k}\Omega$. Consider a current I_1 along the wire with R_1 (going into the top node), I_2 along the wire with R_2 (going out of the top node) and I_3 along the wire with R_L (going out of the top node). Using Kirchhoff's laws and Ohm's law (across the resistors) gives:

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ B_1 - R_1 I_1 - R_L I_3 &= 0 \\ B_2 - R_2 I_2 + R_L I_3 &= 0 \end{aligned}$$

Solving gives

$$\begin{aligned} I_1 &= \frac{(B_1 + B_2)R_{\mathrm{L}} + R_1 B_2}{R_{\mathrm{L}}(R_1 + R_2) + R_1 R_2} = 3\ \mathrm{mA} \\ I_2 &= \frac{(B_1 + B_2)R_{\mathrm{L}} + R_2 B_1}{R_{\mathrm{L}}(R_1 + R_2) + R_1 R_2} = 6\ \mathrm{mA} \\ I_3 &= \frac{B_1}{R_1} = 8\ \mathrm{mA} \end{aligned}$$

To solve for the Thévenin equivalent circuit, we will first find $\epsilon_{\mathrm{eq}} \equiv V_{AB}$. We can the principle of superposition and break the circuit in two: Circuit 1 (Figure 3, where B_1 is "turned on") and Circuit 2 (Figure 4, where B_2 is "turned on"). To solve for $V_{AB} = V_2$ in Circuit 1, we can use the voltage divider equation (Equation 11) giving

$$V_{AB,1} = \left(\frac{R_2}{R_1 + R_2} \right) B_1$$

To solve for $V_{AB} = -V_1$ in Circuit 2, we can use the voltage divider equation (Equation 11) with a negative due to the polarity of the source, giving

$$V_{AB,2} = -\left(\frac{R_1}{R_1 + R_2} \right) B_2$$

When combining the results by superposition ($V_{AB} = V_{AB,1} + V_{AB,2}$) we get

$$\epsilon_{\text{eq}} = \left(\frac{R_2 B_1 - R_1 B_2}{R_1 + R_2} \right) = 6 \text{ V}$$

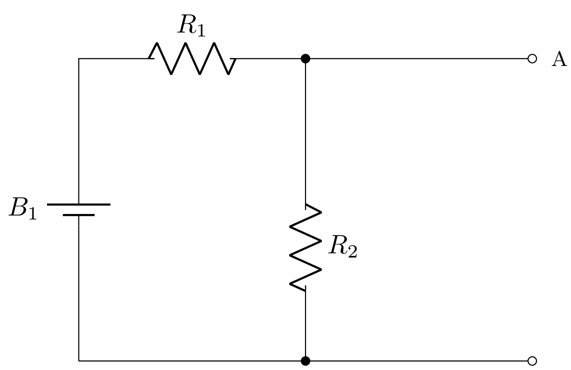


Figure 3: Circuit 1 using the Thévenin method, considering the B_1 to be on and the other sources off.

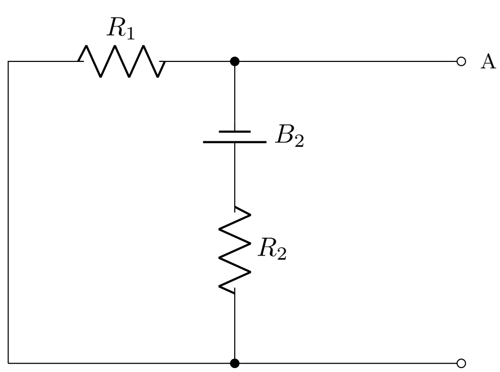


Figure 4: Circuit 2 using the Thévenin method, considering the B_2 to be on and the other sources off.

Since the resistors R_1 and R_2 are in parallel, the equivalent resistance of the Thévenin circuit is

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2} = 1 \text{ k}\Omega$$

2.8 Aside: AC Circuits

So far we have only discussed "direct current" (DC) circuits. In such circuits, current only moves in one direction. There are also "alternating current" (AC) circuits, that have a variable (often sinusoidal) current. These circuits are what the power grid is based on and therefore is what comes out of your wall (even though most household electronics convert this power to DC before using). AC circuits change dynamically in time and so are a bit more complicated and involve solving differential equations. It is common to Fourier transform to solve such equations, making it not as complicated. We will not discuss such circuits here, but if you take further physics or electrical engineering courses, you will certainly see them in some circuits course.

References

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