Ph 1b Recitation Notes
Section 7

Jaeden Bardati
jbardati@caltech.edu

January 11, 2024

1 Introduction to Special Relativity

1.1 Einstein's Postulates

There are two fundamental postulates that give rise to special relativity. They are

1. The laws of physics are the same in all inertial reference frames.[1]
2. Light propagates through empty space with a definite speed c, independent of the speed of the observer.[2]

These postulates lead directly to a few interesting phenomena:

1. Time dilation
2. Length contraction
3. Relativity of simultaneity

These will eventually be combined using Lorentz transformations. For now though, we will start with understanding time dilation.

[1] Inertial reference frames are defined as frames which are moving at constant velocity with respect to another inertial frame. You can tell that you are in a non-inertial frame by observing a failure in Newton's First Law (objects that are in motion stay in motion in a straight line, unless acted on by a force). Note that this is more of a definition than a physical law. In fact, it is not true in general when you consider curved spacetime geodesics (ie. general relativity).

[2] Note that the second postulate is really somewhat redundant since it follows from the first postulate combined with Maxwell's equations (which describe electromagnetism). More generally, if you have a physical law that describes a speed of light c in a given inertial frame, then it must also be constant for all inertial frames by the first postulate. For historical and pedagogical reasons however, it is still considered a postulate of special relativity. In reality, there are other fundamental assumptions hidden in SR such as spatial homogeneity and isotropy.

1.2 Time Dilation

Consider a box of sidelength a moving with constant velocity v in the +x direction, relative to the ground. Both the box and the ground are therefore inertial frames: According to the ground, the box moves at speed v in the +x direction and according to the box, the ground moves at a speed v in the -x direction. Denote the frame of the ground with primes and the frame of the box without primes.

Now, imagine we emit a light beam directly upwards (+y direction) in the box. Thus, the total distance the light travels in the box frame is a. According to the ground though, this light beam leans to one side by an amount $v\Delta t'$ (where $\Delta t'$ is the time it takes for the light to reach the top of the box, in the ground frame), due to the motion of the box relative to the ground. Thus, the total distance the light travels in the ground frame is $\sqrt{a^2 + (v\Delta t')^2}$. By postulate 2, the speed of light is the same in both frames ($c = c'$), so $\sqrt{a^2 + (v\Delta t')^2} = c\Delta t'$ and $a = c\Delta t$ (where Δt is the time it takes for the light to reach the top of the box, in the box frame). Then, it only takes a bit of algebra to show that

$$\Delta t' = \left(\frac{1}{\sqrt{1 - \frac{v}{c}^2}} \right) \Delta t \tag{1}$$

This is the equation for time dilation. We can better understand this if we consider the Lorentz factor $\gamma \equiv \left(\frac{1}{\sqrt{1 - \frac{v}{c}^2}} \right)$ and consider the limits. In the classical limit $\frac{v}{c} \ll 1$, $\gamma \approx 1$, so the two time differences are equal, as expected. In the relativistic limit $v \to c$, $\gamma \to \infty$, so the time difference in the frame of the ground grows very large.[3] This effectively means that, under the postulates that Einstein proposed, time is dilated when seen from a moving frame.

There are a few things we can notice about equation 1:

> First, you need a very large v to create any significant effect on γ (and thus the time dilation factor). Even if $v = 0.5c$, $\gamma \approx 1.15$. This is good for the everyday world (we can almost always forget about this), but bad for designing experiments (at least in the early 1900s).
>
> Second, there seems to be a sort of preferred frame for a given object[4]: the frame in which the object is at rest. We call this the "proper frame". This causes the time to be minimal, and as we will see with length contraction, the length to be maximal.

[3] Note that this kind of approximation is very useful to

[4] Though, not preferred enough to break the first postulate.

A final observation we might make is that it seems it may be convenient to use what are called “relativistic units”. These are units where the value of the speed of light is $c = 1$. This allows us to simply write $\gamma = \left(\frac{1}{\sqrt{1-v^2}}\right)$, and allows us to easily use units of distance such as “light-hours” or “light-years”, etc.

1.3 Length Contraction

Similar to time dilation is length contraction. We can once again define a proper frame for length (as will will soon see). Important differences are that length contraction is *maximal* when in the proper frame, whereas time dilation is *minimal* when in the proper frame. We will briefly cover the regular derivation of the length contraction formula.

Consider a rod of (proper) length l. Consider also a clock that moves at speed v in the direction of the length of the rod, relative to the rod. Relative to the rod, the clock will take a time $t = \frac{l}{v}$ to get from one end of the rod to the other. Due to time dilation, the clock (if set to $t' = 0$ initially) will read $t' = (l/v)\sqrt{1 - v^2}$ after it reaches the end of the rod. In the clock's frame, the rod moves backwards at a speed v, so $t' = \frac{l'}{v}$. Therefore, we get the length contraction formula

$$l' = l\sqrt{1 - \frac{v}{c}^2} \tag{2}$$

We can once again use the Lorentz factor γ to reduce the equation down to $l = \gamma l'$. Again, l here is the proper length (ie. its maximal length possible, and length when at rest).

A crucial point to stress is that length contraction only occurs *along the direction of motion*. Therefore, if the rod were perpendicular to the motion, it would not contract at all (assuming an infinitesimally thick rod). Interestingly, if you move at relativistic speeds parallel to some component of a rod that has a component in a perpendicular direction, then the rod will appear to change angle.

1.4 Relativity of Simultaneity

A concept that is often held dearly by our physical intuition is the concept of simultaneity being true for all frames. Namely, if two events are simultaneous in one frame, then they should be simultaneous to all frames. This is completely false in the relativistic regime.

Consider two stars (A and B) some astronomical distance from each other, both at rest with each other. Let now, from the perspective of the center point of the two stars, both stars go supernovae at the same time. Recall that, since c is finite, light takes time to reach an observer. Thus, what is really happening is that light is taking a finite time to reach the observer from both stars. If we now displace the observer to be very close to the star A, it will see the star A go supernovae before it sees the star B explode.[5] This does not quite break simultaneity yet, since we could just adjust for the time it takes for the light travel and still maintain the belief that events are inherently occurring at same time.

Now, let's say the observer is once again centered in between the stars, but moves towards the star A at a relativistic speed v. For a stationary observer (ie. $v = 0$), the light (by definition) should reach the observer at the same time.[6] For a moving observer (ie. $v \neq 0$), this is not the case. To understand this, consider the finite (constant) speed of light. Within the time it takes for the light from star A to reach the observer's original distance, the observer has already passed it since it is moving in that direction. So, the light from star A requires a shorter amount of time to arrive at the observer. Similarly, the observer will have moved away from star B, making the light from star B require a longer time to arrive at the observer. Thus, an observer moving towards star A will see it explode first, before star B. Thus, simultaneity is frame-dependent.

If we track the mathematics through this argument (See Helliwell [2010] section 6.3 for the details), we would find that: If we synchronize clocks A and B at rest and measure a proper length l between them, then an observer moving at speed v towards A will observe it have a time

$$t_A = \frac{vl}{c^2} \tag{3}$$

when $t_B = 0$. When using this equation in problem sets, take care to properly analogue the situation.

1.5 Problem Solving Strategies in SR

Many problems from special relativity arise from a misuse of frames in equations. A straightforward strategy to combat this is to go through a few basic steps:

1. Define the frames you will use

2. List what you know and what you do not know, in each frame

[5] If, of course, the observer does not first die from the intense radiation from being right next to a supernova.

[6] We call the supernovae "synchronized" in this frame.

3. Find unknowns using appropriate frame changes and physical relations within frames.

It is important that you only use relations such as $v = \frac{\Delta x}{\Delta t}$, within a single frame. Then, you can use time dilation, length contraction, etc.[7] to transform the quantity to the target frame. A useful way to keep track of this is to denote frames with primes or tildes. For example, t may be the time as observed by observer O and t' is the same quantity but as observed by observer O'.

Also, consider carefully what is and what is not a proper frame. Namely, a proper frame (or "rest frame") is the frame in which a measured quantity is at rest. This implies, through the time dilation and length contraction formulae that times measured in the rest frame (called "proper time") are minimal and that lengths measured in the rest frame (called "proper lengths") are maximal.

For quantitative questions and those relating to paradoxes, consider first whether there is relativity of simultaneity. Many so called "paradoxes" arise from forgetting this fact. Moreover, remember that in special relativity, there is no such thing as perfectly rigid objects. This is because, the speed of sound (ie. the natural speed of density perturbations) in the material must necessarily be finite and less than the speed of light.

2 Example: Λ Particle

Problem (Helliwell [2010] prob. 4.9): A Λ particle created in a high-energy collision moved at $v = 0.99c$ in the lab and traveled 55 cm before decaying. What was its lifetime in its own rest frame?

There are two frames to consider: The lab (denote with primes) and the Λ particle (denote without primes). In the frame of the lab, the particle had a lifetime of

$$\Delta t' = \frac{\Delta x'}{v} = \frac{0.55\text{ m}}{0.99\ (3.00 \times 10^8\text{ m/s})} = 1.85 \times 10^{-9}\text{ s}$$

To get this in the frame of the Λ particle, we can use the time dilation equation, where Δt (in the frame of the particle) is the proper time. Thus,

$$\Delta t = \sqrt{1 - \frac{v}{c}^2}\ \Delta t' = \sqrt{1 - (0.99)^2}\ (1.85 \times 10^{-9}\text{ s}) = 2.6 \times 10^{-10}\text{ s}$$

So, the lifetime of the Λ particle in its own rest frame is 2.6×10^{-10} s.

[7] This will eventually be combined into one type of transformation: Lorentz boosts.

3 Example: North Pole

Problem (Helliwell [2010] prob. 4.13): Take two identical clocks. Set one on an ice floe at the North Pole and the other on Isla Santiago, one of the Galapagos Islands, in the Pacific Ocean, right on the equator. The clock at the North Pole is nearly inertial (neglecting the fact that the Earth orbits the Sun, the Sun orbits the galactic nucleus, etc.). A clock at rest on Isla Santiago is not inertial, since it circles Earth's center every 24 hours. According to time dilation, this clock should run slow compared to the clock at the North Pole. What fraction of a second does the Isla Santiago clock lose per day? Note that Earth's radius is 6400 km.

There are two frames: The inertial frame of the North Pole (denote O) and the non-inertial frame of Isla Santiago (denote O'). We can approximate the frame of Isla Santiago as being inertial since the acceleration is very low. Since the radius is $R = 6400$ km and the period of rotation is $T = 24$ hours, we can find the tangential velocity v

$$v = \frac{2\pi R}{T} = \frac{2\pi(6400 \text{ km})}{24 \text{ hours}} \approx 465 \text{ m/s}$$

Then, we can use the time dilation formula with the North Pole as the proper frame to get

$$\Delta t \approx \frac{\Delta t'}{\sqrt{1 - (v/c)^2}}$$

where $\Delta t' = 24$ hours and $v = 465$ m/s. Therefore, the amount that the Isla Santiago clock loses everyday is

$$\Delta t - \Delta t' \approx \Delta t' \left(\frac{1}{\sqrt{1 - (v/c)^2}} - 1 \right)$$

However, typing this in a calculator will not work precisely since this the first term in the parentheses is very close to 1. Therefore, we can approximate using the binomial expansion (generally, $(1 + x)^n \approx 1 + nx$ for small x) and get that $\frac{1}{\sqrt{1-(v/c)^2}} \approx 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2$

$$\Delta t - \Delta t' \approx \frac{\Delta t'}{2}\left(\frac{v}{c}\right)^2 \approx \frac{24 \text{ hours} \times \frac{3600 \text{ s}}{1 \text{ hour}}}{2} \left(\frac{465 \text{ m/s}}{3.00 \times 10^8 \text{ m/s}} \right)^2 \approx 1.04 \times 10^{-7} \text{ s}$$

Therefore, to first order, the amount that the Isla Santiago clock loses per day is 1.04×10^{-7} s.

4 Example: Rogue Planet

Problem (Helliwell [2010] prob. 5.3): A rogue planet from a distant galaxy passes by Earth. If the planet's rest frame diameter is 9,000 km and it passes by a single Earth clock in time 0.04 s, how fast is the planet moving relative to Earth?

Again, there are two frames: The rogue planet (denote without primes) and the Earth (denote with primes).[8] The length of the rogue planet is $l = 9000$ km, according to the rogue planet frame and let it be l' in the Earth frame. The change in time is $\Delta t' = 0.04$ s. We have that $l' \equiv v'\Delta t'$, where v' is the speed of the rogue planet in the frame of Earth. Using the length contraction formula we get that

$$v'\Delta t' = l' = l\sqrt{1 - \left(\frac{v'}{c}\right)^2}$$

With some algebra, we can solve for v'

$$v' = \frac{l}{\sqrt{(\Delta t')^2 + (l/c)^2}} = \frac{9000 \text{ km}}{\sqrt{(0.04 \text{ s})^2 + (9000 \text{ km})^2/(3.00 \times 10^8 \text{ m/s})^2}} = (3/5)c$$

So the speed of the planet relative to Earth is (3/5)c.

5 Example: Synchronized Clocks

Problem (Helliwell [2010] prob. 6.1): Two clocks have been previously synchronized in our frame of reference. We stand beside one and look at the other, which is $d = 30$ m away. (a) What will the other clock appear to read when the clock beside us reads $t = 0$? (b) Now the distant clock is carried to us at the constant speed $v = 30$ m/s. By how much will the two clocks differ when they are side-by-side?

a) To synchronize the clocks, light must travel from one clock to the other. Thus, from our point of view, we are effectively looking backwards in time at the other clock. The other clock will therefore read (according to us) $t = -\frac{d}{c} = -\frac{30 \text{ m}}{3.0\times10^8 \text{ m/s}} = -1.0 \times 10^{-7}$ s.

b) According to us, the far clock is moving at $v = 30$ m/s and traverse a distance of $d = 30$ m, so our clock will change by $\Delta t = \frac{d}{v} = 1$ s. The far clock will observe

[8] Note that my choice of prime frames and non-prime frames is completely arbitrary, but I typically like to keep the non-prime frame as the proper frame for some important quantity. In this case, the length of the rogue planet.

the distance between us as $d' = d\sqrt{1-\left(\frac{v}{c}\right)^2}$ due to length contraction. It will therefore observe itself take a time interval of $\Delta t' = \frac{d'}{v} = \frac{d}{v}\sqrt{1-\left(\frac{v}{c}\right)^2}$ to reach us. Thus, there will be a difference between the clocks of

$$|\Delta t - \Delta t'| = \frac{d}{v}\left(1 - \sqrt{1-\left(\frac{v}{c}\right)^2}\right)$$

Since v is small, we can use the binomial expansion to get

$$|\Delta t - \Delta t'| \approx \frac{d}{v}\left(1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1\right) = \frac{vd}{2c^2} = 5.0 \times 10^{-15}\ \text{s}$$

Thus, there is a difference in the time of the clocks by about 5.0×10^{-15} s.

6 Example: Colliding Elephants

Problem (adapted from Faraoni [2013]): A rigid cage containing two elephants equipped with jet-packs is plummeting straight down towards the ground at relativistic speeds. Relative to the cage, the elephants, aligned vertically, decide to engage their jetpacks for a moment and collide into one another for fun. With perfect elastic collision, they bounce backwards and hit the walls of the cage. In the cage frame, they hit the walls at the same time and no momentum is imparted to the bulk of the cage. However, in the frame of the ground, the top wall hits its elephant before the bottom wall does, due to the additional motion of the cage (relativity of simultaneity). The ground must therefore see a momentary jerk in upwards and then downwards when the elephants hit their walls. Since the cage would experience a change in velocity in the perspective of the ground, but the ground does not change velocity in the perspective of the cage, this seems to violate the principle of relativity. What is going on here?

The issue arises from the fact that perfectly rigid objects do not exist in special relativity. In a material, in order for one end of the object to communicate that it has moved to the other, it must send a density perturbation to the other side, limited by the speed of sound in the material. A perfectly rigid object would therefore have an infinite speed of sound. However, in special relativity, the speed of sound in the material cannot be faster than the speed of light. There must thus be some delay time between when the top wall of the cage propagates the information that it has moved (in the form of a density wave) to the bottom of the cage and vice versa. Thus, the cage will effectively momentarily stretch, not jerk up and down.

References

V. Faraoni. *Special Relativity*. Undergraduate Lecture Notes in Physics. Springer International Publishing, Cham, Switzerland, 2014 edition, Aug. 2013.

T. Helliwell. *Special Relativity*. University Science Books, 2010.