Ph 12b Recitation Notes

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1 Multiple Particles in Quantum Mechanics

Imagine we have now two particles instead of one. They can each have their own state $|u\rangle$ and $|v\rangle$. Then, we mark their combined state as $|u\rangle \otimes |u\rangle$ and otherwise use them as usual, but act on them separately. Similarly, we could have three particles with $|u\rangle \otimes |v\rangle \otimes |w\rangle$, etc.

For example, if I have the first particle in the spin up directional state $|\uparrow\rangle \equiv \left|s = \frac{1}{2}, m = \frac{1}{2}\right\rangle$ and the second particle in the spin down state $|\downarrow\rangle \equiv \left|s = \frac{1}{2}, m = -\frac{1}{2}\right\rangle$ and want to measure the spin of the first particle, I do

$$\hat{S}_z^{(1)} \left(|\uparrow\rangle \otimes |\downarrow\rangle\right) = \frac{\hbar}{2} \left(|\uparrow\rangle \otimes |\downarrow\rangle\right)$$

And

$$\hat{S}_z^{(2)} \left(|\uparrow\rangle \otimes |\downarrow\rangle\right) = -\frac{\hbar}{2} \left(|\uparrow\rangle \otimes |\downarrow\rangle\right)$$

Or even

$$\hat{S}_z^{(1)} \hat{S}_z^{(2)} \left(|\uparrow\rangle \otimes |\downarrow\rangle\right) = -\left(\frac{\hbar}{2}\right)^2 \left(|\uparrow\rangle \otimes |\downarrow\rangle\right)$$

Note that the way that I indicate the particles here inherently assumes the particles are order-able (generally called 'distinguishable"). However, this need not be true! Remember that QM describes how probabilities act *before* measurement, but only when a state is measured does it collapses to a single state. So, if you had two of the same type of particle (e.g. electrons), there is no reason that you would know the difference between them because they could, in principle, exactly change places (and spins, etc.) and no measurable would change. So we need to combine them in a way that is somehow symmetric to their interchange. There are two ways to do this ($|+\rangle$ and $|-\rangle$) with two particles in state $|u\rangle$ and $|v\rangle$, namely

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|u\rangle |v\rangle \pm |v\rangle |u\rangle\right) \tag{1}$$

where if your particles fall under the symmetric case $|+\rangle$, they are called *bosons*, and if they fall under the anti-symmetric case, they are called *fermions*. Note that fermions cannot be put in the same state (since then the wavefunction would not be normalizable), but bosons can! This leads to certain statistical fluids like Bose-Einstein condensates where all the bosons fall into the same state, causing many particles to effectively act like one. Fermions not wanting to be in the same state is the reason that white dwarfs (the remnants of many stars after their deaths) are held up against gravity despite there being no nuclear fusion in its core (which is how regular stars are held up). Fermions have half-integer spin ($\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ...) and bosons have integer spin (0, 1, 2, ...). We will see why.

Suppose now you want to find the total spin of your two particles (with spin direction $|l_1, m_1\rangle$ and $|l_2, m_2\rangle$). Then, you might write this combined operator as the sum of the individual measurements $\hat{S}_z = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$, giving

$$S_z \left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right) = S_z^{(1)} \left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right) + S_z^{(2)} \left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right)$$

$$= \frac{m_1\hbar}{2}\left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right) + \frac{m_2\hbar}{2}\left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right)$$

$$= (m_1 + m_2)\frac{\hbar}{2}\left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right)$$

So the resulting m value is simply $m = m_1 + m_2$. However, s is not so simple. We know that there are four possible combinations of the spin states

$$|\uparrow\rangle \otimes |\uparrow\rangle\,, \quad |\uparrow\rangle \otimes |\downarrow\rangle\,, \quad |\downarrow\rangle \otimes |\uparrow\rangle\,, \quad |\downarrow\rangle \otimes |\downarrow\rangle$$

The first state has $m = \frac{1}{2} + \frac{1}{2} = 1$, the second and third have $m = 0$ and the last has $m = -1$. It seems like $s = s_1 + s_2 = 1$ for this new state, since m ranges from $-s$ to s, but there are two states with $m = 0$. There are two ways of combining them

$$\frac{1}{\sqrt{2}}\big(\,|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle\,\big) \quad \text{or} \quad \frac{1}{\sqrt{2}}\big(\,|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle\,\big)$$

These are called the "uncoupled states." The former of the two states has $m = 1$, but the later of the states has $m = 0$. You can apply a lowering operator $\hat{S}_- = \hat{S}_-^{(2)} + \hat{S}_-^{(2)}$ to the $|\uparrow\rangle \otimes |\uparrow\rangle$ state, to find $\hat{S}_-(|\uparrow\rangle \otimes |\uparrow\rangle) = \hbar(|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle)$. Since the $\hat{S}_-$ does not change the total spin, it seems like the symmetric state has $s = 1$ and the other has $s = 0$ (applying the lowering or raising operator to the antisymmetric state will yield 0). This cleanly separates the combined states into two combinations

$$\begin{cases} |s = 1, m = 1\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \\ |s = 1, m = 0\rangle = \frac{1}{\sqrt{2}}\big(\,|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle\,\big) \\ |s = 1, m = -1\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \end{cases} \qquad \text{(triplet)}$$

and

$$|s = 0, m = 0\rangle = \frac{1}{\sqrt{2}}\big(\,|\downarrow\rangle \otimes |\uparrow\rangle - |\uparrow\rangle \otimes |\downarrow\rangle\,\big) \qquad \text{(singlet)}$$

These are called the "coupled states." You can verify all this by using the Hermitian total spin operator

$$\hat{S}^2 = \left(\hat{\boldsymbol{S}}^{(1)} + \hat{\boldsymbol{S}}^{(2)}\right) \cdot \left(\hat{\boldsymbol{S}}^{(1)} + \hat{\boldsymbol{S}}^{(2)}\right) = \left(\hat{S}^{(1)}\right)^2 + \left(\hat{S}^{(2)}\right)^2 + 2\hat{\boldsymbol{S}}^{(1)} \cdot \hat{\boldsymbol{S}}^{(2)}$$

Thus, combining a general spin s_1 with s_2 will lead to states with spins $s = (s_1 + s_2), (s_1 + s_2 - 1), ..., |s_1 - s_2|$, and the ms are just additive.

2 Recitation Problems

2.1 Quantum Mechanics in a Flat World

(a) Consider a 3D quantum mechanical particle in cylindrical coordinates. Find the ordinary differential equations that govern its wavefunction for a general potential V that depends only on the cylindrical radius s. Solve the ϕ and z equations. *Hint:* Use the cylindrical Laplacian

$$\nabla^2\psi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \psi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (2)$$

Answer: $\psi(s, \phi, z) = S(s)e^{i(m_\phi \phi + m_z z/s)}$ with the equation

$$s^2\frac{\partial^2 S}{\partial s^2} + s\frac{\partial S}{\partial s} + \left(\frac{2ms^2}{\hbar}\left[E - V(s)\right] - m_\phi^2 - m_z^2\right)S(s) = 0 \qquad (3)$$

(b) From now on, we will consider a "flat world" where the particle is confined to a circular flat sheet (set $m_z = 0$). Let primes denote s derivatives, $\tilde{E} \equiv \frac{2mE}{\hbar}$ and $\tilde{V}(s) \equiv \frac{2mV(s)}{\hbar}$ such that the equation you want to solve is written simply as

$$s^2 S'' + sS' + \left(\left[\tilde{E} - \tilde{V}(s)\right] s^2 - m_\phi^2\right) S(s) = 0 \tag{4}$$

Consider a free particle and find the resulting wavefunction confined within a circle of radius R. No need to normalize. *Hint:* Use Bessel functions of the first and second kind ($J_m(x)$ and $N_m(x)$, respectively) which are the two solutions of $x^2y'' + xy' + (x^2 - m^2)f = 0$.

Look up the behaviour of Bessel functions and explain why we need not worry about solutions of the second kind. Find the total energy if j_{mp} is the pth zero of the first Bessel function. Is this consistent with our understanding of free particles in Cartesian coordinates as $e^{i\vec{k}\cdot\vec{r}}$? Justify using the Jacobi-Anger expansion

$$e^{ix\cos\theta} \equiv \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\theta} \tag{5}$$

Answer: Solution is

$$\psi(s,\phi) \propto J_{m_\phi}(k_{mp}s)e^{im_\phi \phi} \tag{6}$$

with energies

$$E_{nmp} = \frac{\hbar^2 j_{mp}^2}{2mR^2} \tag{7}$$

This is indeed consistent with the solution in Cartesian coordinates. No Bessel functions of the second kind since it blows up as $s \to 0$. There is a nice visual of the cylindrically-confined particle at https://demonstrations.wolfram.com/QuantumMechanicalParticleInACylinder/

2.2 Spin and Clebsch-Gordan Coefficients

(a) Is the singlet state symmetric or anti-symmetric under exchange of the spin-$\frac{1}{2}$ particles? Does this match with your understanding of fermions having $\frac{1}{2}$ spin? What about the triple state?

(b) The reason for the discrepancy in the triplet state is that we are missing the position part of the state, we only have spin. If we add the position, what does this imply about the symmetry/anti-symmetry of the position state in the singlet and triplet states?

(c) What we have done above with the two spin particles is find the Clebsch-Gordan Coefficients $C^{s_1 s_1 s}_{m_1 m_2 m}$ for $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, $m_1 = \pm 1$, $m_2 = \pm 1$, $s = 0, 1$ and $m = 0, \pm 1$. These coefficients are defined such that

$$|s, m\rangle = \sum_{m_1 + m_2 = m} C^{s_1 s_1 s}_{m_1 m_2 m} \left(|s_1, m_1\rangle \otimes |s_2, m_2\rangle\right) \tag{8}$$

or equivalently

$$|s_1, m_1\rangle \otimes |s_2, m_2\rangle = \sum_s C^{s_1 s_1 s}_{m_1 m_2 m} \,|s, m = m_1 + m_2\rangle \tag{9}$$

They are useful to use when combining states, and there are tables of these available.

From now on, consider the addition of a boson with spin $s_1 = 1$ and a fermion with spin $s_2 = \frac{1}{2}$ List all possible combinations of s and m that result from these two particles.

(d) Use lowering operators on the highest m states for each s to derive all the coupled states in terms of the uncoupled states. With these, state the relevant Clebsch-Gordan coefficients. Verify with a lookup table.