Ph 12b Recitation Notes

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1 Quantum Mechanics Review

1.1 Generalized Uncertainty Principle

In a general sense, the usual Heisenberg uncertainty principle

$$\sigma_p \sigma_z \geq \frac{\hbar}{2} \tag{1}$$

comes from the fact that a wavepacket cannot simultaneously have a well-defined wavelength (which is related to its momentum through $\lambda = \frac{h}{p}$) and a well-defined position at the same time.

We can generalize this to any two operators $\hat{A}$ and $\hat{B}$ where

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right)^2 \tag{2}$$

It is clear that operators that commute are mutually observable. Those that do not are called **incompatible** observables (e.g. x and p or E and t).

From this, we can find the generalized Ehrenfest theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle \left[\hat{H}, \hat{Q} \right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \tag{3}$$

We can also find an energy-time uncertainty relation

$$\sigma_E \sigma_t \geq \frac{\hbar}{2} \tag{4}$$

This means that there is a fundamental limit how precise the energy of something is based on your measurement timescale.

1.2 Quantum Mechanics in 3D

We can also re-write the Schrödinger equation in three dimensions. Namely,

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{x}) \Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{5}$$

where ∇^2 is the Laplacian ($\vec{\nabla} \cdot \vec{\nabla}$). Notice that the Hamiltonian generalizes is still $\hat{H} = \frac{p^2}{2m} + V(\vec{x})$, but now the momentum operator is $\vec{p} = -i\hbar\vec{\nabla}$.

Notice that we can still solve separation of variables, but now we need to include three functions (one for each variable, (e.g. $\Psi(x, y, z) = X(x)Y(y)Z(z)$), and we will separate the single partial differential equation into three ordinary differential equations. Otherwise, the same general process holds.

2 Recitation Problems

2.1 Uncertainty Relation's Effect on Line Broadening

The Rydberg formula for hydrogenic atoms is:

$$\frac{1}{\lambda} = R_H Z^2 \left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \tag{6}$$

where Z is the atomic number, $R_H = 1.097 \times 10^5$ cm^{-1}. Consider a light-emitting gas emits a Hydrogen $n = 2$ to $n = 1$ transition photon every ~ 1 nanoseconds.

(a) Estimate the intrinsic width of the observed spectral line. You can assume that $\Delta\lambda \ll \lambda_0$ and use the fact that $E = h\nu$ and $c = \lambda\nu$ for light. Recall $\hbar \equiv \frac{h}{2\pi} \approx 1.055 \times 10^{-27}$ erg s.

(b) Compare this with an estimate of the Doppler broadening on the line $\Delta\nu = \left(\frac{\Delta v}{c}\right)\nu_0$ if the material is turbulently mixing with RMS speed $\sim 10^4$ km/s. Also, compare the thermal broadening width of the line $\sqrt{\frac{k_B T}{mc^2}}\nu_0$ with $k_B = 1.38 \times 10^{-16}$ erg/K, assuming a gas temperature of $T \sim 10^4$ K. What can you conclude about the dominant broadening effect in this system? At what temperatures and turbulent velocities would intrinsic broadening become relevant?[1]

2.2 Three Dimensional Momentum and Position Commutation Relations

Find the commutation relations between $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$, where $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$. Also show that the three dimensional momentum operator $\vec{p} = i\hbar\vec{\nabla}$ satisfies $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$.

2.3 3D Potential Problem

Solve the infinite cubical well. Namely, the 3D potential where

$$V(x, y, z) = \begin{cases} 0 & \text{if all } 0 < x, y, z < a \\ \infty & \text{otherwise} \end{cases} \tag{7}$$

Hint: Use separation of variables in Cartesian coordinates.

2.4 Griffiths Problem 6.30 (time permitting)

[1] For those who are curious, this is the Lyman-α line, which is vital to astronomy. Here, we ask whether we need to consider the natural width of this line when it is broadened through a process such as rapid 10^4 km/s rotation around a supermassive black hole or thermal broadening with a gas temperature $T \sim 10^4$ K. Spectral lines such as these play a pivotal role in identifying and characterizing rapidly accreting supermassive black holes in distant galaxies.