

# Ph 12b Recitation Notes

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Please see the previous weeks' notes for details on connecting quantum mechanics to linear algebra. In this recitation, we will focus on applications.

## 1 Recitation Problems

### 1.1 Custom problem from last week (continued)

The time translation operator is defined as

$$\hat{U}(t) = \exp\left(-\frac{it}{\hbar}\hat{H}\right) \quad (1)$$

(a) Show that  $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$  satisfies the time-dependent Schrödinger equation. *Hint:* Use a Taylor series expansion of the exponential if you are not familiar with using operators in the exponent. Assume that the Hamiltonian does not depend on time.

*Solution:* Time-dependent Schrödinger equation is

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

but  $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\left(e^{-it\hat{H}/\hbar}\right)|\psi(0)\rangle = i\hbar\frac{\partial}{\partial t}\left(-\frac{i}{\hbar}\hat{H}e^{-it\hat{H}/\hbar}\right)|\psi(0)\rangle = \hat{H}\left(e^{-it\hat{H}/\hbar}\right)|\psi(0)\rangle = \hat{H}|\psi(t)\rangle$ .  
QED.

(b) Show that the time translation operator is unitary. *Hint:* Remember that  $\hat{H}$  is hermitian (that is  $\hat{H}^\dagger = \hat{H}$ ). You can use the small  $\hat{H}$  approximation and Taylor expand to first-order if you prefer.

*Solution:* We have

$$\begin{aligned} \hat{U}(t)^\dagger &= \left(e^{-it\hat{H}/\hbar}\right)^\dagger \\ &= \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\hat{H}t\right)^n\right]^\dagger \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar}\hat{H}^\dagger t\right)^n \\ &= e^{it\hat{H}^\dagger/\hbar} \\ &= e^{it\hat{H}/\hbar} \end{aligned}$$

But also the inverse of  $U$  is

$$\hat{U}(t)^{-1} = \left( e^{-it\hat{H}/\hbar} \right)^{-1} = e^{it\hat{H}/\hbar}$$

Thus,  $\hat{U}(t)^\dagger = \hat{U}(t)^{-1}$  so the time translation operator is indeed unitary.

(c) <sup>1</sup>So far, we have been using the so-called “Schrödinger picture,” where the states (wave functions) evolve with time, but operators are generally time-independent (with the recent exception of this time-translation operator). In the “Heisenberg picture,” operators are generally time-dependent and states are not. We write operators in such a picture as  $\hat{Q}_H(t) \equiv \hat{U}(t)^\dagger \hat{Q} \hat{U}(t)$ . Show that both pictures are completely consistent with each other by differentiating this equation to get the Heisenberg equations of motion

$$i\hbar \frac{d}{dt} \hat{Q}_H(t) = [\hat{Q}_H(t), \hat{H}] \quad (2)$$

and then plugging in the time translation operator above to get the usual time-dependent Schrödinger equation with  $|\Psi(t)\rangle = U(t)|\Psi_H\rangle$ .

(d) Plug in  $\hat{Q} = \hat{x}$  and  $\hat{Q} = \hat{p}$  to get differential equations for  $\hat{x}_H$  and  $\hat{p}_H$  in the Heisenberg picture for a single particle with mass  $m$  in a potential  $V(x)$ .

## 1.2 Sakurai 1.6 and 1.8 (modified)

Using the rules of bra-ket algebra, prove or evaluate the following.

(a)  $(\hat{X}\hat{Y})^\dagger = \hat{Y}^\dagger\hat{X}^\dagger$ , where  $\hat{X}$  and  $\hat{Y}$  are general operators. *Hint:* Expand in an orthonormal basis.

(b) Suppose  $|i\rangle$  and  $|j\rangle$  are eigenvectors of some Hermitian operator  $\hat{A}$ . Under what conditions can we conclude that  $|i\rangle + |j\rangle$  is also an eigenvalue of  $\hat{A}$ ? Justify.

## 1.3 Griffiths 6.30 (time permitting)

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<sup>1</sup>Last week we got this far (part c). We will begin our problems for this week here.