Ph 12b Recitation Notes

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1 The Language of Quantum Mechanics: Linear Algebra

1.1 Inner products (repeated)

We can define an **inner product** $\langle x|y\rangle$ of two vectors $|x\rangle$ and $|y\rangle$ in some vector space as satisfying the following properties:

1. **Conjugate symmetry**: $\langle x|y\rangle = \overline{\langle y|x\rangle}$
2. **Linearity**: $\langle ax + by|z\rangle = a\langle x|z\rangle + b\langle y|z\rangle$
3. **Positive-definite**: $\langle x|x\rangle > 0$ if $x \neq 0$

In principle, this can be defined anyway you want, so long as it satisfies the above properties. However, for our use we will most just have two forms. Namely, the usual form you are used to whenever vectors are represented by a list of numbers $|a\rangle = A_1|1\rangle + A_2|2\rangle + ... + A_N|N\rangle$ and $|b\rangle = B_1|1\rangle + B_2|2\rangle + ... + B_N|N\rangle$ for some basis $\{|1\rangle, |2\rangle, ..., |N\rangle\}$, giving

$$\langle a|b\rangle = A_0^* B_0 + A_1^* B_1 + ... \tag{1}$$

And, also one that may be new to you which is for functions $f(x)$ and $g(x)$ in some vector space we can define the inner product as

$$\langle f|g\rangle = \int_a^b f(x)^* g(x) dx \tag{2}$$

Indeed, we can often represent functions *themselves* as vectors in some space with an infinite-dimensional basis. This basis could be many things. For example, we know from calculus that we can represent any smooth[1] function within some range by an infinite series of polynomials (think Taylor series). This means that $\{1, x, x^2, ...\}$ is a basis of the vector space containing all smooth functions. Moreover, we know from Fourier analysis that we can equally represent all smooth functions by an infinite series of sinusodial functions, so $\{\sin x, \cos x, ...\}$ is also a basis. The above inner product allows us to multiply these vectors. This is particularly important to quantum mechanics since we are often dealing with an functions rather than N-tuples of numbers. Such vector spaces are called **Hilbert spaces**. Now, just a bit more linear algebra theory before connecting it back to QM.

The **Schwarz inequality** is

$$|\langle f|g\rangle| \leq \sqrt{\langle f|f\rangle \langle g|g\rangle} \tag{3}$$

A vector $|f\rangle$ is **normalized** if $\langle f|f\rangle = 1$ and two vectors $|f\rangle$ and $|g\rangle$ are **orthogonal** if $\langle f|g\rangle = 0$. We say a set of vectors $\{f_n\}$ are **orthonormal** if $\langle f_m|f_n\rangle = \delta_{mn}$ where δ_{mn} is the Kronecker delta (returns 1 if $m = n$ and 0 otherwise).

[1] Smooth in this context means that all infinite-derivatives are continuous.

1.2 Dual vectors

We know that vectors are expressed with $|v\rangle$ (so called "kets", from brac*ket*) and inner products are expressed with $\langle u|v\rangle$. You might wonder if we can take just the first part, i.e. $\langle u|$ (called "bras", from *bra*cket[2]). In fact, we can and they are called *dual vectors*.

Broadly, dual vectors act on vectors and return scalars. So, the way to think about them almost closer to an operator than a vector in some sense (they are *not* commutative with vectors). In particular, they act on vectors to give the innerproduct. That is

$$\langle u| \; |v\rangle = \langle u|v\rangle \tag{4}$$

But using them in this way can be quite useful sometimes, for example when distributing

$$(\langle u| + \langle v|)\,|v\rangle = \langle u|v\rangle + \langle v|v\rangle \tag{5}$$

However, dual vectors are more than just a notational trick, it turns out since they are related by the *adjoint*. Namely,

$$\langle u|^* = |u\rangle \quad \text{and} \quad |u\rangle^* = \langle u| \tag{6}$$

Thus, dual vectors encode the information about both the vector and the inner product in them. As we will see later, they become particularly useful when dealing with projections.

1.3 Quantum States as Vectors

As we saw above, we can represent functions as vectors in a vector space if we like. We know that the quantum state is represented by the wavefunction and that these wavefunctions must be normalized. It turns out that we can form a vector space of just the square-integrable functions. Practically, this means that the wavefunction Ψ can be represented as a vector where $\langle\Psi|\Psi\rangle = 1$ means it is normalized. Then, if we have some basis of states $|\psi_n\rangle$, then we can represent any wavefunction as $\langle\psi|\psi\rangle = \sum_{n=1}^{\infty} c_n\psi_n$ for an appropriate choice of c_ns. By Fourier transforms, we can also show that these coefficients are $c_n = \langle\psi_n|\psi\rangle$. You may notice that this is very similar to what we have been doing with stationary states.

In fact, in this language, the time-independent Schrödinger equation becomes $\bar{H}\,|\psi\rangle = E\,|\psi\rangle$. Indeed, we can now interpret finding the energy of a solution we are simply solving for the *eigenvalues* of the Hamiltonian. We also note that the stationary states are the *eigenvalues* of the Hamiltonian.

In this language, expectation values of an operator $\hat{Q}$ are simply $\left\langle \Psi \middle| \hat{Q}\Psi \right\rangle$.[3] Indeed, observables are represented with **hermitian** operators: operators $\hat{Q}$ that satisfy $\left\langle \psi_1 \middle| \hat{Q}\psi_2 \right\rangle = \left\langle \hat{Q}\psi_1 \middle| \psi_2 \right\rangle$, for all ψ_1 and ψ_2. The value of the eigenvalue . We can find, for instance, that the Hamiltonian is Hermitian and since its eigenvalue is the energy, then energy is an observable. We also call the eigenvalues of a Hermitian operator **determinate** states.

We call the eigenbasis (set of eigenvalues) of an operator its **spectrum**. If the spectrum of an operator is *discrete*, then the eigenfunctions are normalizable (and thus are physically realizable states). If the spectrum is *continous*, then they are not normalizable. The best that can happen is instead Dirac orthonormality where $\langle f_p|f_{p'}\rangle = \delta(p - p')$.

One can find a general form of the uncertainty principle between two operators where

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right)^2 \tag{7}$$

[2] No idea what happened to the "c".

[3] Note that this is sometimes written as $\langle\Psi|\hat{Q}|\Psi\rangle$.

It is clear that operators that commute are mutually observable. Those that do not are called **incompatible** observables (e.g. x and p or E and t).

From this we can find the generalized Ehrenfest theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{Q}\right]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle \tag{8}$$

This is particularly useful for finding expectation values just from the operators.

We can change from a basis $\{|e_n\rangle\}$ to a basis $\{|\alpha\rangle, |\beta\rangle, ...\}$ with $|\alpha\rangle = \sum_n \langle e_n|\alpha\rangle |e_n\rangle$, $|\beta\rangle = \sum_n \langle e_n|\beta\rangle |e_n\rangle$, and so on. This can be nicely captured with the **projection operator** $\hat{P} \equiv \sum_n |e_n\rangle\langle e_n|$ (so long as $\{|e_n\rangle\}$ is orthonormal, and turns to an integral if it is a Dirac orthonormal continuous basis).

Recall that the commutator of operators A and B is defined as

$$[A,B] = AB - BA \tag{9}$$

And the anti-commutator is

$$\{A,B\} = AB + BA \tag{10}$$

2 Recitation Problems

2.1 Custom problem

Imagine you have a particle with three orthogonal states $|-1\rangle$, $|0\rangle$ and $|1\rangle$.

(a) Construct the projection operator.

Answer: $\hat{P} = |-1\rangle\langle -1| + |0\rangle\langle 0| + |2\rangle\langle 2|$

(b) Now, use the projection operator to find the coefficients of $|v\rangle = \frac{1}{\sqrt{3}}|-1\rangle + \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$ in the new basis $\{|\psi_n\rangle\}$ where you know that:

$$\langle -1|\psi_0\rangle = 0, \quad \langle -1|\psi_1\rangle = -\frac{1}{\sqrt{2}}, \quad \langle -1|\psi_2\rangle = \frac{1}{\sqrt{2}}, \tag{11}$$
$$\langle 0|\psi_0\rangle = -\frac{1}{\sqrt{2}}, \quad \langle 0|\psi_1\rangle = 0, \quad \langle 0|\psi_2\rangle = -\frac{1}{\sqrt{2}}, \tag{12}$$
$$\langle 1|\psi_0\rangle = \frac{1}{\sqrt{2}}, \quad \langle 1|\psi_1\rangle = -\frac{1}{\sqrt{2}}, \quad \langle 1|\psi_2\rangle = 0, \tag{13}$$
$$\tag{14}$$

and all other inner products are zero. Is the state normalized in the $\{|-1\rangle, |0\rangle, |1\rangle\}$ basis? Is it normalized in the $\{|\psi_n\rangle\}$ basis? If it is not or if it were not normalized, what does that mean about the orthonormality of the $\{|\psi_n\rangle\}$ basis?

2.2 Sakurai Problem 1.2

Show that

$$[AB, CD] = -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB \tag{15}$$