Ph 12b Recitation Notes

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1 The Language of Quantum Mechanics: Linear Algebra

There is a nice way of representing quantum mechanics that we can represent using linear algebra. First, let's review the basics.

1.1 Linear Algebra Basics

A linear vector space is defined as a set V of vectors $|v\rangle$ coupled with addition and vector-scalar multiplication operations, with the following properties (let $|u\rangle$, $|v\rangle$, and $|w\rangle$ be vectors in V and α, β be scalars in e.g. $\mathbb{C}$):

1. The set V is **closed** under linear combinations (combinations of vector addition and vector-scalar multiplication). This means that if $|u\rangle$ and $|v\rangle$ are in V, then a linear combination of them $(\alpha\,|u\rangle + \beta\,|v\rangle)$ is also in V.
2. Vector addition is **associative**: $(|u\rangle + |v\rangle) + |w\rangle = |u\rangle + (|v\rangle + |w\rangle)$
3. Vector addition is **commutative**: $(|u\rangle + |v\rangle) = |v\rangle + |u\rangle$
4. There exists a unique **identity vector** $|0\rangle$ such that: $|u\rangle + |0\rangle = |u\rangle$
5. Every vector $|v\rangle$ has an **additive inverse** $-|v\rangle$ such that: $|v\rangle + -|v\rangle = |0\rangle$
6. Vector-scalar multiplication is **associative**: $\alpha(\beta\,|u\rangle) = (\alpha\beta)\,|u\rangle$
7. Vector-scalar multiplication is **distributive** over addition in the regular ways: $\alpha(|u\rangle + |v\rangle) = \alpha\,|u\rangle + \alpha\,|v\rangle$ and $(\alpha + \beta)\,|u\rangle = \alpha\,|u\rangle + \beta\,|u\rangle$

For example, 3-dimensional real space $\mathbb{R}$ forms a vector space. Note that the set V has infinite elements even if we have a finite basis of vectors. However, we can also have an infinite number of basis vectors. This is the case for many situations in quantum mechanics.

Note that you can also define scalars more generally than just complex or real numbers too (e.g. in the context of a mathematical *field*), but we don't really have to worry about that for our use here. Now, on to a few more definitions.

A set of vectors $\{|v_n\rangle\}$ is **linearly independent** if we cannot get the identity vector through some nontrivial linear combination of all of them. Namely, if $\sum_{n=0} \alpha_n\,|v_n\rangle = |0\rangle$ where not all $\alpha_n = 0$.

If a vector space is N-dimensional, then any set of N linearly independent vectors forms a **basis**. These basis are important because they allow you to form any other vector in the space with a countable number of vectors.

We can also define an **inner product** $\langle x|y\rangle$ of two vectors $|x\rangle$ and $|y\rangle$ in some vector space as satisfying the following properties:

1. **Conjugate symmetry**: $\langle x|y \rangle = \overline{\langle y|x \rangle}$

2. **Linearity**: $\langle ax + by|z \rangle = a \langle x|z \rangle + b \langle y|z \rangle$

3. **Positive-definite**: $\langle x|x \rangle > 0$ if $x \neq 0$

In principle, this can be defined anyway you want, so long as it satisfies the above properties. However, for our use we will most just have two forms. Namely, the usual form you are used to whenever vectors are represented by a list of numbers $|a\rangle = A_1 |1\rangle + A_2 |2\rangle + ... + A_N |N\rangle$ and $|b\rangle = B_1 |1\rangle + B_2 |2\rangle + ... + B_N |N\rangle$ for some basis $\{|1\rangle, |2\rangle, ..., |N\rangle\}$, giving

$$\langle a|b \rangle = A_0^* B_0 + A_1^* B_1 + ... \tag{1}$$

And, also one that may be new to you which is for functions $f(x)$ and $g(x)$ in some vector space we can define the inner product as

$$\langle f|g \rangle = \int_a^b f(x)^* g(x) dx \tag{2}$$

Indeed, we can often represent functions *themselves* as vectors in some space with an infinite-dimensional basis. This basis could be many things. For example, we know from calculus that we can represent any smooth[1] function within some range by an infinite series of polynomials (think Taylor series). This means that $\{1, x, x^2, ...\}$ is a basis of the vector space containing all smooth functions. Moreover, we know from Fourier analysis that we can equally represent all smooth functions by an infinite series of sinusodial functions, so $\{\sin x, \cos x, ...\}$ is also a basis. The above inner product allows us to multiply these vectors. This is particularly important to quantum mechanics since we are often dealing with an functions rather than N-tuples of numbers. Such vector spaces are called **Hilbert spaces**. Now, just a bit more linear algebra theory before connecting it back to QM.

The **Schwarz inequality** is

$$|\langle f|g \rangle| \leq \sqrt{\langle f|f \rangle \langle g|g \rangle} \tag{3}$$

A vector $|f\rangle$ is **normalized** if $\langle f|f \rangle = 1$ and two vectors $|f\rangle$ and $|g\rangle$ are **orthogonal** if $\langle f|g \rangle = 0$. We say a set of vectors $\{f_n\}$ are **orthonormal** if $\langle f_m|f_n \rangle = \delta_{mn}$ where δ_{mn} is the Kronecker delta (returns 1 if $m = n$ and 0 otherwise).

1.2 Quantum States as Vectors

As we saw above, we can represent functions as vectors in a vector space if we like. We know that the quantum state is represented by the wavefunction and that these wavefunctions must be normalized. It turns out that we can form a vector space of just the square-integrable functions. Practically, this means that the wavefunction Ψ can be represented as a vector where $\langle \Psi|\Psi \rangle = 1$ means it is normalized. Then, if we have some basis of states $|\psi_n\rangle$, then we can represent any wavefunction as $\langle \psi|\psi \rangle = \sum_{n=1}^{\infty} c_n \psi_n$ for an appropriate choice of c_ns. By Fourier transforms, we can also show that these coefficients are $c_n = \langle \psi_n|\psi \rangle$. You may notice that this is very similar to what we have been doing with stationary states.

In fact, in this language, the time-independent Schrödinger equation becomes $\bar{H} |\psi\rangle = E |\psi\rangle$. Indeed, we can now interpret finding the energy of a solution we are simply solving for the *eigenvalues* of the Hamiltonian. We also note that the stationary states are the *eigenvalues* of the Hamiltonian.

In this language, expectation values of an operator $\hat{Q}$ are simply $\left\langle \Psi \middle| \hat{Q}\Psi \right\rangle$.[2] Indeed, observables are represented with **hermitian** operators: operators $\hat{Q}$ that satisfy $\left\langle \psi_1 \middle| \hat{Q}\psi_2 \right\rangle = \left\langle \hat{Q}\psi_1 \middle| \psi_2 \right\rangle$, for all ψ_1 and ψ_2. The value of the eigenvalue . We can find, for instance, that the Hamiltonian is Hermitian and since its

[1] Smooth in this context means that all infinite-derivatives are continuous.

[2] Note that this is sometimes written as $\langle \Psi|\hat{Q}|\Psi \rangle$.

eigenvalue is the energy, then energy is an observable. We also call the eigenvalues of a Hermitian operator **determinate** states.

We call the eigenbasis (set of eigenvalues) of an operator its **spectrum**. If the spectrum of an operator is *discrete*, then the eigenfunctions are normalizable (and thus are physically realizable states). If the spectrum is *continous*, then they are not normalizable. The best that can happen is instead Dirac orthonormality where $\langle f_p | f_{p'} \rangle = \delta(p - p')$.

One can find a general form of the uncertainty principle between two operators where

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right)^2 \tag{4}$$

It is clear that operators that commute are mutually observable. Those that do not are called **incompatible** observables (e.g. x and p or E and t).

From this we can find the generalized Ehrenfest theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle \left[\hat{H}, \hat{Q} \right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \tag{5}$$

This is particularly useful for finding expectation values just from the operators.

We can change from a basis $\{|e_n\rangle\}$ to a basis $\{|\alpha\rangle, |\beta\rangle, ...\}$ with $|\alpha\rangle = \sum_n \langle e_n|\alpha\rangle |e_n\rangle$, $|\beta\rangle = \sum_n \langle e_n|\beta\rangle |e_n\rangle$, and so on. This can be nicely captured with the **projection operator** $\hat{P} \equiv \sum_n |e_n\rangle \langle e_n|$ (so long as $\{|e_n\rangle\}$ is orthonormal, and turns to an integral if it is a Dirac orthonormal continuous basis).

2 Recitation Problems

2.1 Custom Conceptual Exercise

We have learnt that set require a few conditions to be considered vector spaces. Namely, vectors must be closed under linear combinations, have an identity vector, and inverse for all elements, that vector addition is associative and commutative and that vector scale multiplication is associative and distributive in the usual way.

a) Give an example of a process that is non-commutative but associative.

Answer: An example is matrix multiplication: $AB \neq BA$, but $(AB)C = A(BC)$.

b) Give an example of a process that is non-commutative and non-associative.

Answer: An example is the vector cross product. It is non-commutative since $\hat{x} \times \hat{y} = \hat{z}$ but $\hat{y} \times \hat{x} = -\hat{z}$. It is non-associative since $\hat{x} \times (\hat{x} \times \hat{y}) = \hat{x} \times \hat{z} = -\hat{y}$, but $(\hat{x} \times \hat{x}) \times \hat{y} = \hat{0} \times \hat{y} = \hat{0}$.

c) Give an example of a process that is commutative and non-associative.

Answer: An example is a process that returns the winner from rock paper scissors. Doing rock vs paper gives the same result as doing paper vs rock, so it is commutative. It is non-associative since doing rock vs paper first (returning paper) vs scissors, returns scissors, whereas paper vs scissors first (returning scissors) vs rock, returns rock.

d) Does the set of positive real numbers $\mathbb{R}^+$ form a vector space under the usual addition and multiplication? Why or why not?

Answer: No, since there is no additive inverse.

e) Does the set of positive real numbers excluding zero $\mathbb{R}^{\nvDash}\backslash\{\vec{0}\}$ form a vector space under the usual addition and multiplication? Why or why not?

Answer: No, since there is no identity.

f) Do angles $\phi \in [0, 2\pi]$ form a vector space under the usual addition and multiplication? Why or why not?

Answer: No, since it is not closed.

g) If we modify the last example such that we still consider angles $\phi \in [0, 2\pi]$ but now use modular addition and multiplication (the regular addition/multiplication mod 2π), does this form a vector space? Why or why not?

Answer: No, since it does not obey the distributive law $c \odot (\phi_1 \oplus \phi_2) \neq c \odot \phi_1 \oplus c \odot \phi_2$. A counterexample is $c = \frac{1}{2}$ and $\phi_1 = \phi_2 = \frac{3\pi}{2}$, the left side yields $\frac{\pi}{2}$ but the right yields $\frac{3\pi}{2}$.

2.2 Griffiths Problem 3.1

2.3 Griffiths Problem 3.2

2.4 Griffiths Problem 3.3