Physics 1c Practical
Spring 2024

Comprehensive Special Relativity Problem

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1. **Relativistic Elephants (10 pts).**

   A cubical cage containing two elephants is plummeting straight down towards the ground at speed $v = 0.25c$ (relative to the ground). The elephants, aligned vertically, initially travel at a constant velocity towards each other and collide into one another for fun. With perfect elastic collision in the center of the cage, they bounce backwards exactly vertically at equal speed $u = 0.50c$, relative to the center of mass of the cage (call this the cage frame). Recall that $c = 3.00 \times 10^8$ m/s, ignore air friction and assume particle-like elephants.

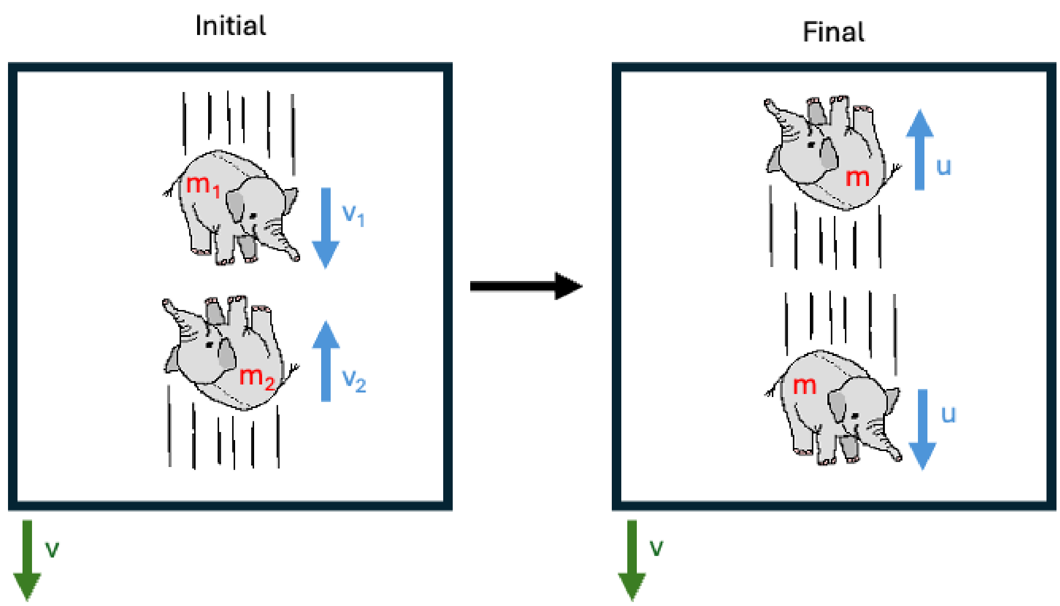

Figure 1: An illustration of the collision of elephants in a cage.

   (a) If the cage has a rest vertical length of $l_0 = 25$ m, what is its length as measured by an observer on the ground after collision? What is the cage length as measured by the elephants in the final state? Does the cage's horizontal length change at all in any of the inertial frames? Why or why not?

Length contraction. Relative to the ground, the cage has length

$$l = l_0\sqrt{1 - v^2/c^2} = 24.2 \text{ m}$$

Relative to the elephants, the cage has length

$$l_{\text{top}} = l_0\sqrt{1 - u^2/c^2} = 21.7 \text{ m}$$

The horizontal lengths of the elephants do not change since there is no relative velocity between any of the frames in that direction.

(b) What is the final velocity of each elephant relative to the ground?

Velocity addition. In the y direction,

$$v'_{\text{top}} = \frac{u - v}{1 - vu/c^2} = 0.29c \;\text{ and }\; v'_{\text{bot}} = \frac{-u - v}{1 + vu/c^2} = -0.67c$$

No component in the other directions.

(c) The top elephant has an initial rest mass of $m_1 = 3000$ kg and the bottom elephant has a mass of $m_2 = 4000$ kg. During the instant of collision, the elephants remember that they are both made of highly reactive material. This means that mass is not conserved, causing both elephants to each have a final mass of m. If the top elephant had an initial speed of $v_1 = 0.40c$, what was the initial speed of the other elephant v_2, relative to the cage? What is m?

Conservation of relativistic 4-momentum.

$$m_1\gamma_1 v_1 - m_2\gamma_2 v_2 = -mu\gamma_u + mu\gamma_u = 0$$

$$m_1\gamma_1 + m_2\gamma_2 = m\gamma_u + m\gamma_u = 2m\gamma_u$$

Solving this system of equations gives $v_2 = 0.31c$ and $m = 3200$ kg.

(d) What are the initial 4-velocities of the elephants, relative to the cage frame? Use a Lorentz transformation to also find this relative to the ground, and find the 3-velocity in that frame. Do the same for the final 4-velocities. Does this match your answer for problem 1b?

The initial 4-velocities of the elephants, relative to the cage frame, are

$$u^{\nu}_{\text{top,i}} = \gamma_1(c, \boldsymbol{v_1}) = (1.091, 0, -0.436, 0)c \quad \text{and} \quad u^{\nu}_{\text{bot,i}} = \gamma_2(c, \boldsymbol{v_2}) = (1.052, 0, 0.327, 0)c$$

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \gamma & 0 & -\frac{v}{c}\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{v}{c}\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.033 & 0 & -0.258 & 0 \\ 0 & 1 & 0 & 0 \\ -0.258 & 0 & 1.033 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies u^{\mu'}_{\text{top,i}} = (1.24, 0, -0.73, 0)c \quad \text{and} \quad u^{\mu'}_{\text{bot,i}} = (1.002, 0, 0.066, 0)c$$

The 3-velocities, relative to the ground, are therefore $v'_{\text{top,i}} = -0.73c/1.24 = -0.59c$ and $v'_{\text{bot,i}} = 0.066c/1.002 = -0.07c$. Similarly, the final 4-velocities are

$$u^{\nu}_{\text{top,f}} = (1.155, 0, 0.577, 0)c \quad \text{and} \quad u^{\nu}_{\text{bot,f}} = (1.155, 0, -0.577, 0)c$$
$$\implies u^{\mu'}_{\text{top,f}} = (1.044, 0, 0.298, 0)c \quad \text{and} \quad u^{\mu'}_{\text{bot,f}} = (1.342, 0, -0.894, 0)c$$

So, relative to the ground, $v'_{\text{top,f}} = 0.298c/1.044 = 0.29c$ and $v'_{\text{bot,f}} = -0.894c/1.342 = -0.67c$, which indeed agrees with problem 1b.

(e) Some time after they collide with each other, the elephants will hit the cage walls at the exact same time relative to the cage frame. However, in the frame of the ground, the top elephant hits the top wall before the bottom elephant hits the bottom wall. The top elephant measured the cage to have a side length of $L = 25$ m during its travel to the wall. How long does does it take, after the top elephant has hit the top wall, for the bottom elephant to hit its wall, relative to the ground?

Relativity of simultaneity. The rest length of the wall is

$$L_0 = \frac{L}{\sqrt{1 - u^2/c^2}} = 28.9 \text{ m}$$

The time difference is therefore

$$\Delta t = \frac{vL_0}{c^2} = 24 \text{ ns}$$

(f) In the center of mass frame of the cage, the elephants hit the walls at the same time and thus no total momentum is imparted to the bulk of the cage (since they now have equal mass and velocity). Since, in the frame of the ground, the top elephant hits its wall before the bottom one does, the ground observer should see a momentary jerk upwards and then downwards when the elephants hit their respective walls. This seems to break the principle of relativity! Explain in a couple sentences what the resolution of this paradox is. What is the key assumption made here that is broken and is thus disallowed by special relativity?

The paradox arises from the fact that perfectly rigid objects do not exist in special relativity. In a material, in order for one end of the object to communicate that it has moved to the other, it must send a density perturbation to the other side, limited by the speed of sound in the material. A perfectly rigid object would therefore have an infinite speed of sound. However, in special relativity, the speed of sound in the material cannot be faster than the speed of light. There must thus be some delay time between when the top wall of the cage propagates the information that it has moved (in the form of a density wave) to the bottom of the cage and vice versa. Thus, the cage will not jerk up and down, but rather stretch.

(g) Elephant 1 did not know the answer to the last problem, and is therefore worried about having possibly broken special relativity. So, on its return directly towards Elephant 2 (after their possibly non-elastic and/or reactive collisions with the walls), it shines a laser at Elephant 2. The laser is set to shine at a wavelength of $\lambda = 770$ nm, according to Elephant 1. Elephant 2 sees this laser light at a wavelength of $\lambda' = 385$ nm. Knowing this, what can Elephant 1 conclude about their relative speed? Has Elephant 1 been reassured that they are, at least, travelling less than the speed of light? Show that this experiment should never give a speed larger than that of light, regardless of λ and λ'.

Relativistic Doppler shift.

$$\lambda' = \lambda\sqrt{\frac{1-w/c}{1+w/c}} \implies w = \left(\frac{(\lambda/\lambda')^2-1}{(\lambda/\lambda')^2+1}\right)c = 0.6c$$

Indeed, they are travelling less than the speed of light. Moreover, their relative speed approaches $\pm c$ as $\lambda/\lambda' \to \infty$ or zero, implying their relative speed is bounded by the speed of light.